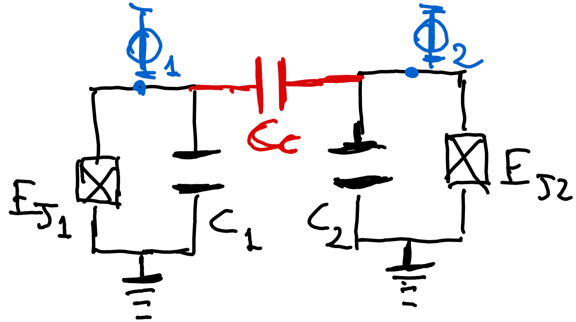
TWO QUBIT GATE



$$H_1 = 4E_{C_1} n_1^2 - E_{J1}\cos\varphi_1$$

$$H_2 = 4E_{C_2} n_2^2 - E_{J2}\cos\varphi_2$$

$$H_c = 4E_{C_{12}} n_1 n_2$$

Derivation

$$\mathcal{L} = \frac{1}{2}C_1\dot{\Phi}_1^2 + \frac{1}{2}C_2\dot{\Phi}_2^2 + \frac{1}{2}C_c\left(\dot{\Phi}_1 - \dot{\Phi}_2\right)^2 +$$

$$+ E_{J1}\cos\left[2\pi\frac{\Phi_1}{\Phi_0}\right] + E_{J2}\cos\left[2\pi\frac{\Phi_2}{\Phi_0}\right]$$

We can calculate the conjugate variables to Φ_1, Φ_2:

$$Q_1 = \frac{\partial\mathcal{L}}{\partial\dot{\Phi}_1} = [C_1 + C_c]\dot{\Phi}_1 - C_c\dot{\Phi}_2$$

$$Q_2 = \frac{\partial\mathcal{L}}{\partial\dot{\Phi}_2} = [C_2 + C_c]\dot{\Phi}_2 - C_c\dot{\Phi}_1$$

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} C_1 + C_c & -C_c \\ -C_c & C_2 + C_c \end{bmatrix} \begin{bmatrix} \dot{\Phi}_1 \\ \dot{\Phi}_2 \end{bmatrix}$$

$\underline{Q}$ $\qquad$ $\underline{\underline{C}}$ capacitance matrix

$$\underline{\underline{C}}^{-1} = \frac{1}{(C_1+C_c)(C_2+C_c)+C_c^2} \begin{bmatrix} C_2 + C_c & C_c \\ C_c & C_1 + C_c \end{bmatrix}$$

$$H = Q_1 \dot{\Phi}_1 + Q_2 \dot{\Phi}_2 - \mathcal{L}$$

$$\mathcal{L} = \frac{1}{2} \underline{Q}^T \underbrace{\underline{\underline{C}}^{-1} \underline{Q}} - E_{J1} \cos \frac{2\pi}{\Phi_0} \Phi_1 - E_{J2} \cos \frac{2\pi}{\Phi_0} \Phi_2 =$$

$$\begin{bmatrix} C_2 + C_c & C_c \\ C_c & C_1 + C_c \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} =$$

$$\begin{bmatrix} Q_1 , Q_2 \end{bmatrix} \begin{bmatrix} (C_2+C_c) Q_1 + C_c Q_2 \\ C_c Q_1 + (C_1 + C_c) Q_2 \end{bmatrix}$$

$$H = \frac{1}{2}\frac{1}{C_1 + C_c + \frac{C_c^2}{C_2 + C_c}} Q_1^2 + \frac{1}{2}\frac{1}{C_2 + C_c + \frac{C_c^2}{C_1 + C_c}} Q_2^2 +$$

$$+ \frac{2 C_c}{(C_1 + C_c)(C_2 + C_c) + C_c^2} Q_1 Q_2 +$$

$$- E_{J1} \cos \frac{2\pi}{\Phi_0} \Phi_1 - E_{J2} \cos \frac{2\pi}{\Phi_0} \Phi_2$$

- Now we can promote the classical variables to quantum operators:

$$Q_i \longrightarrow 2e\hat{n}_i$$

$$\Phi_i \longrightarrow \frac{\Phi_0}{2\pi}\hat{\varphi}$$

$$E_{C_1} = \frac{e^2}{C_1 + C_c + \frac{C_c^2}{C_2 + C_c}}$$

$$E_{C_2} = \frac{e^2}{C_2 + C_c + \frac{C_c^2}{C_1 + C_c}}$$

$$E_{C_{12}} = \frac{2 C_c e^2}{(C_1 + C_c)(C_2 + C_c) + C_c^2}$$

$$H_1 = 4E_{C_1} n_1^2 - E_{J_1}\cos\varphi_1 \simeq -\frac{1}{2}\hbar\omega_1\sigma_{z1}$$

$$H_2 = 4E_{C_2} n_2^2 - E_{J2}\cos\varphi_2 \simeq -\frac{1}{2}\hbar\omega_2\sigma_{z2}$$

$$H_c = 4E_{C_{12}} n_1 n_2 = 4E_{C12}\sum_{i,j,k,l} \underbrace{\langle ij|n_1 n_2|kl\rangle}_{\langle i|n_1|k\rangle\cdot\langle j|n_2|l\rangle} \overbrace{|ij\rangle\langle kl|}^{|i\rangle\langle k|\otimes|j\rangle\langle k|}$$

$$\underbrace{\langle i|n_1|k\rangle}_{\substack{\langle 0|n_1|1\rangle \\ \langle 1|n_1|0\rangle \\ = g_1}} \cdot \underbrace{\langle j|n_2|l\rangle}_{\substack{\langle 0|n_2|1\rangle \\ \langle 1|n_2|0\rangle \\ = g_2}}$$

$$= \underbrace{4E_{C_{12}} g_1 g_2}_{\hbar g}\Big[\left(|0\rangle\langle 1|\right)_1 \otimes \left(|0\rangle\langle 1|\right)_2 + \left(|0\rangle\langle 1|\right)_1 \otimes \left(|1\rangle\langle 0|\right)_2 + \left(|1\rangle\langle 0|\right)_1 \otimes \left(|0\rangle\langle 1|\right)_2 + \left(|1\rangle\langle 0|\right)_1 \otimes \left(|1\rangle\langle 0|\right)_2\Big]$$

$$\sigma_{x_1}\sigma_{x_2} = \Big[|0\rangle\langle 1| + |1\rangle\langle 0|\Big] \otimes \Big[|0\rangle\langle 1| + |1\rangle\langle 0|\Big]$$

$$H = -\frac{1}{2}\hbar\omega_1\sigma_{z1} - \frac{1}{2}\hbar\omega_2\sigma_{z_2} + \hbar g\,\sigma_{x_1}\sigma_{x_2}$$

We are going into the rotating frame of the qubits + we will apply the Rotating Wave Approximation

$$U = \exp[-i\omega_1 t\, \sigma_{z1}/2] \otimes \exp[-i\omega_2 t\, \sigma_{z2}/2]$$

and remembering $\sigma_+ = \sigma_x + i\sigma_y$; $\sigma_- = \sigma_x - i\sigma_y$

We get: $\tilde{H} = -\frac{1}{4}\hbar g\left[e^{-i(\omega_1-\omega_2)}\sigma_{+1}\sigma_{-2} + e^{+i(\omega_1-\omega_2)}\sigma_{-1}\sigma_{+2}\right]$

If $\omega_1 = \omega_2$ [ON RESONANCE]

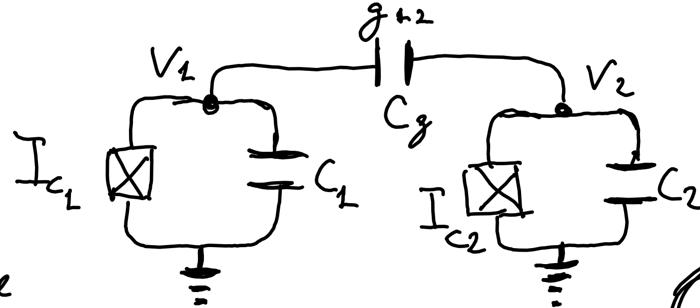
$$\tilde{H} = -\frac{1}{2}\hbar g\left[\sigma_{x1}\sigma_{x2} + \sigma_{y1}\sigma_{y2}\right]$$

To generate entanglement between individual quantum systems it is necessary to engineer an interaction Hamiltonian that connects degrees of freedom in those individual systems.

We will discuss some physical coupling mechanisms.

The Hamiltonian of two coupled systems takes a generic form $H = H_1 + H_2 + H_{int}$, where H_1, H_2 denote the Hamiltonians of the individual quantum systems. The last term, H_{int}, is the interaction Hamiltonian which couples the variables of both systems.

In superconducting circuits, the physical form of the coupling energy is either an electric or magnetic field.



To achieve a capacitive coupling, a capacitor is placed between the voltage nodes of the two participating circuits, yielding an interaction Hamiltonian $H_{int} = C_g V_1 V_2$, where C_g is the coupling capacitance and V_1 (V_2) is the voltage operator of the corresponding voltage node being connected.

Circuit quantization in the limit of $C_g \ll C_1, C_2$ yields

$$H = \sum_{i=1,2} \left[4 E_{C_i} n_i^2 - E_{J,i} \cos(\phi_i) \right] + 4e^2 \frac{C_g}{C_1 C_2} n_1 n_2$$

Hamiltonians individual qubits

$V_i = (2e/C_i)\, n_i$

Regardless of its physical realization, the effect of a coupling on system dynamics is determined by its form as represented in the eigenbasis of the individual systems.

Using second quantization, the system Hamiltonian can be expressed as

$$H = \sum_{i \in 1,2} \left[\omega_i a_i^{+} a_i + \frac{\alpha_i}{2} a_i^{+} a_i^{+} a_i a_i \right] - g \left(a_1 - a_1^{+} \right)\left(a_2 - a_2^{+} \right)$$

where the expression within brackets represent the Duffing oscillator Hamiltonian for the qubits and g is the coupling energy. Since we define $V \propto n \propto i\,(a - a^{+})$, $I \propto \phi \propto (a + a^{+})$ the original "$n_1 n_2$" term becomes: $n_1 n_2 \propto (a_1 - a_1^{+})(a_2 - a_2^{+})$

Such coupling is called "transverse", because the coupling Hamiltonian has nonzero matrix elements only at off-diagonal positions with respect to both oscillators

$${}_i\langle k | a_i - a_i^{+} | k \rangle_i = 0 \quad ; \quad {}_i\langle k \pm 1 | a_i - a_i^{+} | k \rangle_i \neq 0$$

If we can ignore higher energy levels $(k \geq 2)$, because of sufficient anharmonicity of the qubit, we may truncate to

$$H = \sum_{i \in 1,2} \frac{1}{2} \omega_i \sigma_{z,i} + g\, \sigma_{y,1} \sigma_{y,2}$$

This is equivalent at the Hamiltonian of two spins coupled by an exchange interaction. Such a Hamiltonian is most commonly used in contemporary implementations and can generate various types of two-qubit entangling gates.

When both capacitive and inductive couplings are present in the system, both $\sigma_x \sigma_x$ and $\sigma_y \sigma_y$ terms will be present.

So, the interaction term between two capacitively coupled qubits (in the two-level approximation) is given by $H_{qq} = g\, \sigma_{y1} \otimes \sigma_{y2}$, where

$$g_{q-q} = \frac{1}{2}\sqrt{\omega_{q1}\omega_{q2}}\ \frac{C_{q-q}}{\sqrt{C_{q-q}+C_1}\sqrt{C_{qq}+C_2}},$$

with C_{q-q} the qubit-qubit coupling capacitance and C_i the capacitance of each qubit.

We will assume that a direct capacitive coupling between qubits, which are flux-tunable transmon type $\omega_{q_i} \longrightarrow \omega_{q_i}(\Phi_i)$.

We can rewrite the Hamiltonian as

$$H_{qq} = -g\left([\sigma^+ - \sigma^-] \otimes [\sigma^+ - \sigma^-]\right),$$

Using the rotating wave approximation, we arrive at

$$H_{qq} = g\left(e^{i\delta\omega_{12}t}\sigma^+\sigma^- + e^{-i\delta\omega_{12}t}\sigma^-\sigma^+\right),$$

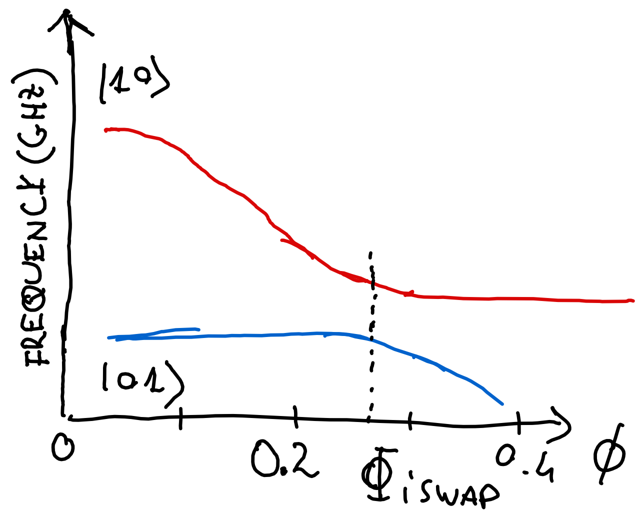
where $\delta\omega_{12} = \omega_{q_1} - \omega_{q_2}$.

If we now change the flux of qubit 1 to bring it into resonance with qubit 2 ($\omega_{q_1} = \omega_{q_2}$), then

$$H_{qq} = g\left(\sigma^+\sigma^- + \sigma^-\sigma^+\right) = \frac{g}{2}\left(\sigma_{xx} + \sigma_{yy}\right)$$

swapping of excitation between the 2 qubits

"XY interaction"



The unitary time evolution operator corresponding to a XY (swap) interaction is

$$U_{qq}(t) = \exp\left[-i\frac{g}{2}(\sigma_x\sigma_x + \sigma_y\sigma_y)t\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(gt) & -i\sin(gt) & 0 \\ 0 & -i\sin(gt) & \cos(gt) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since the qubits are tunable in frequency, we can ~~now~~ consider the effect of tuning the qubits into resonance for a time "$t' = \pi/2g$"

$$U_{qq}\left(\frac{\pi}{2g}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \equiv i\,\text{SWAP}$$

From this result, we see that a capacitive coupling between qubits turned-on for a time $t' \propto 1/g$ leads to implementing a so called "iSWAP" gate, which acts to swap an excitation between the two qubits, and add a phase of $i = e^{i\pi/2}$.

For completeness, we note that for $t'' = \pi/4g$ the resulting unitary

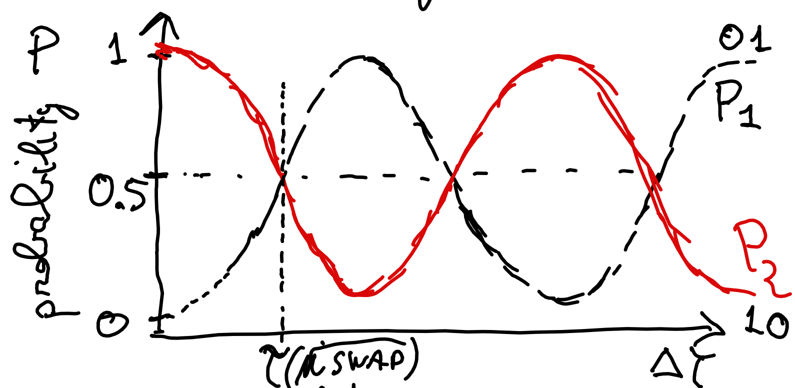
$$U_{gg}\left(\frac{\pi}{4g}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & -i/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \equiv \sqrt{i\text{SWAP}}$$

is typically referred as the "squareroot-iSWAP" gate.

The $\sqrt{i\text{SWAP}}$ gate can be used to generate Bell-like superposition states, e.g. $|01\rangle + i|10\rangle$.

To elucidate the operating principle behind an iSWAP implementation, we can consider the spectrum of a flux-tunable qubit using typical transmon-like parameters. The iSWAP gate is performed at the avoided crossing where $\Phi = \Phi_{i\text{SWAP}}$.

By preparing the qubit 1 in state $|1\rangle$, moving "fast" into the avoided crossing, wait there for a time τ, the excitation is swapped back and forth between the two qubits.



The excitation oscillates back and forth between $|01\rangle$ and $|10\rangle$ with the predicted time "$t' = \pi/2g$". In turn, the frequency of the oscillation can be used to extract the strength of the coupling, $2/t' = g/\pi$. During the flux-tuning process there will be some single qubit phases acquired, given by

$$\theta_z = \int_0^{\tau} dt\,(\omega_q - \omega(t)).$$

This phases should be taken into account or removed. Typical values of qubit coupling via a direct capacitor are 5-40 MHz.

The "iSWAP" cannot generate a CNOT gate by itself.

To implement a CNOT gate, the following sequence will be required (by using iSWAP)